Introduction to topology optimization

Applied and computational mathematics seminar

Mathematics Department
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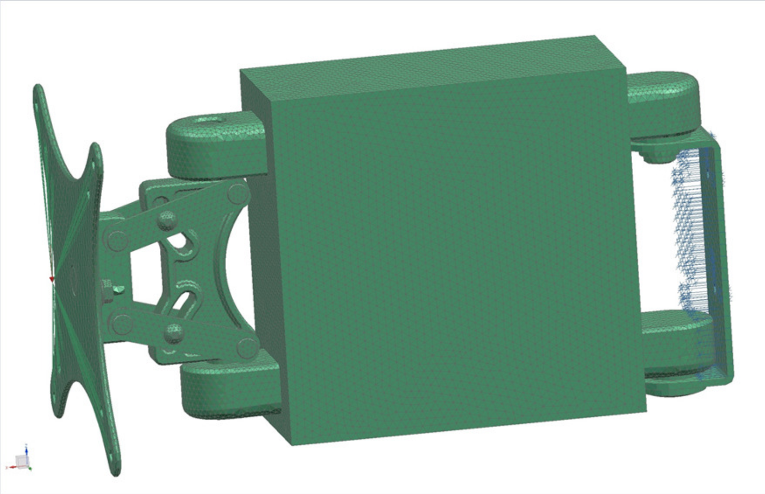
Topics of presentation

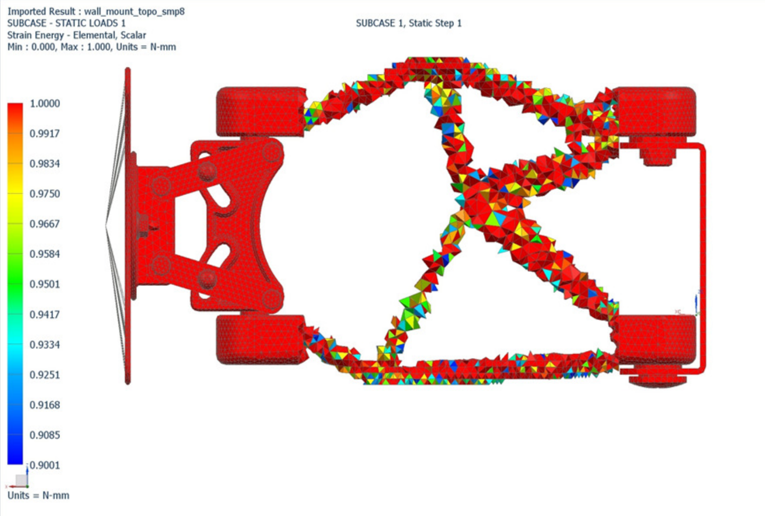
Mathematical foundation

- Concept of topology optimization
- Finite element foundation
- Objectives and constraints
- Matrix sensitivities
- Solution gradients
- Workflow of optimization

Application examples

- Demonstration example
- Cantilever structure
- 4 legged stool
- Drilling platform
- Building bridges
- Additive manufacturing





Topoplogy optimization concept

Global kinetic energy

$$E_k = \frac{1}{2} \iiint_V \rho \dot{q}^2 dV = \frac{1}{2} \iiint_V \dot{q}^T N^T \rho N \dot{q} dV$$

$$E_k = \frac{1}{2} \sum_{i=1}^{N} \dot{q}_i^T \iiint_{V_e} N^T \rho N dV_e \dot{q}_i = \frac{1}{2} \sum_{i=1}^{N} \dot{q}^T M_i \dot{q}$$

Global potential energy

$$E_s = \frac{1}{2} \iiint_V E \varepsilon^2 dV = \frac{1}{2} \iiint_V q^T B^T E B q dV$$

$$E_s = \frac{1}{2} \sum_{i=1}^{N} q_i^T \iiint_{V_e} B^T E B dV_e q_i = \frac{1}{2} \sum_{i=1}^{N} q_i^T K_i q_i$$

Element energies

$$\varepsilon_i = q_i^T K_i q_i ; \kappa_i = \dot{q}_i^T M_i \dot{q}_i$$

Eliminate or decrease density of elements when

$$\varepsilon_i \leq \varepsilon_{\min} ; \kappa_i \geq \kappa_{\min}$$

Topology optimization foundation

Finite element model of structure

$$K = \sum_{i=1}^{N} K_i \; ; \; M = \sum_{i=1}^{N} M_i$$

Element density as design variables

$$d_{\min} \le d_i \le d_{\max} \; ; i = 1, \ldots, N$$

Updated finite element matrices

$$K = \sum_{i=1}^{N} K_i^0 d_i ; K = K_i^0 d_i + \sum_{k=1, k \ne i}^{N} K_k^0 d_k$$

$$M = \sum_{i=1}^{N} M_i^0 d_i ; M = M_i^0 d_i + \sum_{k=1, k \ne i}^{N} M_k^0 d_k$$

Sensitivity of matrices

$$\frac{\partial K}{\partial d_i} = \frac{\partial K_i}{\partial d_i} = K_i^0 = \frac{K_i}{d_i}$$

$$\frac{\partial M}{\partial d_i} = \frac{\partial M_i}{\partial d_i} = M_i^0 = \frac{M_i}{d_i}$$

Topology optimization problem

Minimize an objective function $f = f(K, M)$

subject to one or more constraints $g_j;\ j = 1, 2, \ldots m$

Main objective = minimize weight of structure $W < W_0$

Limit static deformation at a given location $u_j \le u_{\max}$

Increase structural compliance in static solution $C \ge C_{\min}$

Natural frequency magnitude above excitations $\lambda_i \ge \lambda_{\min}$

Weight objective sensitivity

Global gravity vector
$$a_G(i) = [. \quad a_G(i_1) \quad . \quad a_G(i_2) \quad . \quad a_G(i_3) \quad . \quad a_G(i_4) \quad .]^T$$

Weight of the structure
$$W = \sum_{i=1}^{N} W_i$$

Updated weight
$$W(d_i) = d_i W_i^0 + \sum_{k=1, k \neq i}^{N} W_k \, d_k$$

Mass basis
$$W(d_i M) = (M_i^0 d_i + \sum_{k=1, k \neq i}^{N} M_k^0 d_k) a_G$$

Sensitivity of weight
$$\frac{\partial W}{\partial d_i} = M_i^0 a_G(i) = \frac{M_i}{d_i} a_G(i)$$

Gravity load
$$F_g = W; \frac{\partial F_g}{\partial d_i} \neq 0$$

Static deformation sensitivity

Linear static solution

$$Ku = F \qquad u^T K = (K^T u)^T = (Ku)^T = F^T$$

Solution sensitivity

$$\frac{\partial K}{\partial d_i} u + K \frac{\partial u}{\partial d_i} = \frac{\partial F}{\partial d_i}$$

Constant loading force

$$\frac{\partial F}{\partial d_i} = 0 \qquad u^T K \frac{\partial u}{\partial d_i} = -u^T \frac{\partial K}{\partial d_i} u$$

Substitute and premultiply

$$F^T \frac{\partial u}{\partial d_i} = -u^T \frac{\partial K}{\partial d_i} u = -u^T \frac{K_i}{d_i} u$$

Element strain energy

$$\varepsilon_i = u^T K_i u$$

Deformation sensitivity at node j

$$\frac{\partial u(j)}{\partial d_i} = -\frac{1}{F(j) d_i} \varepsilon_i$$

Structural compliance sensitivity

Structural compliance definition $C = F^T u$

Sensitivity of compliance
$$\frac{\partial C}{\partial d_i} = \frac{\partial F^T}{\partial d_i} u + F^T \frac{\partial u}{\partial d_i}$$

Static solution sensitivity
$$\frac{\partial K}{\partial d_i} u + K \frac{\partial u}{\partial d_i} u = \frac{\partial F}{\partial d_i} \quad \frac{\partial u}{\partial d_i} = K^{-1}\left(\frac{\partial F}{\partial d_i} - \frac{\partial K}{\partial d_i} u\right)$$

Compliance in general
$$\frac{\partial C}{\partial d_i} = \frac{\partial F^T}{\partial d_i} u + F^T K^{-1}\left(\frac{\partial F}{\partial d_i} - \frac{\partial K}{\partial d_i} u\right)$$
$$\frac{\partial C}{\partial d_i} = \frac{\partial F^T}{\partial d_i} u + u^T \frac{\partial F}{\partial d_i} - u^T \frac{\partial K}{\partial d_i} u$$

Constant force specific
$$\frac{\partial C}{\partial d_i} = -u^T \frac{\partial K}{\partial d_i} u = -\frac{1}{d_i} \varepsilon_i$$

Natural frequency sensitivity

Eigenvalue analysis solution $$(K - \lambda M)\varphi = 0$$

Eigenvalue solution sensitivity $$\frac{\partial K}{\partial d_i}\varphi + K\frac{\partial \varphi}{\partial d_i} = \lambda\frac{\partial M}{\partial d_i}\varphi + \frac{\partial \lambda}{\partial d_i}M\varphi + \lambda M\frac{\partial \varphi}{\partial d_i}$$

Assumptions $$\varphi^T M \varphi = I \qquad \frac{\partial \varphi}{\partial d_i} = 0$$

Eigenvalue sensitivity $$\frac{\partial \lambda}{\partial d_i} = \varphi^T(\frac{\partial K}{\partial d_i} - \lambda\frac{\partial M}{\partial d_i})\varphi$$

With matrix sensitivities $$\frac{\partial \lambda}{\partial d_i} = \varphi^T\frac{K_i}{d_i}\varphi - \lambda\varphi^T\frac{M_i}{d_i}\varphi = \frac{1}{d_i}(\varepsilon_i - \kappa_i)$$

Element kinetic energy $$\kappa_i = \dot{\varphi}^T M_i\dot{\varphi}; \dot{\varphi} = \lambda\varphi$$

Optimization problem definition

Seeking for the objective $$f(\bar{d}^*) = extremum$$

Subject to constraints $$g_j(\bar{d}^*) \leq g_j^{\max}; \; j = 1, \ldots m$$

Design variables $$\bar{d}^T = \begin{bmatrix} d_1 & \ldots & d_N \end{bmatrix}$$

Objective sensitivities $$\bar{f}^T = \begin{bmatrix} \frac{\partial f}{\partial d_1} & \ldots & \frac{\partial f}{\partial d_N} \end{bmatrix}$$

Constraints sensitivities $$G^T = \begin{bmatrix} \ldots & \ldots & \ldots \\ \frac{\partial g_j}{\partial d_1} & \ldots & \frac{\partial g_j}{\partial d_N} \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Topology optimization flow

Start analysis job

- Set current design variables $\bar{d}^{k} = \left[d_1^k \quad . \quad d_N^k \right]^T ; k = 1, 2, ..$
- Modifiy element matrices $K_i^k = K_i^0 d_i^k ; M_i^k = M_i^0 d_i^k$
- Update assembled model $K^k = \sum_{i=1}^{N} K_i^k ; M^k = \sum_{i=1}^{N} M_i^k$
- Solve analysis problem $K^k u = F \qquad (K^k + \lambda M^k)\varphi = 0$
- Collect and compute $\varepsilon_i^k, \kappa_i^k ; f(\bar{d}^k), g_j(\bar{d}^k); \bar{f}(\bar{d}^k), G(\bar{d}^k)$
- Call optimizer to receive $\bar{d}^{k+1}, \bar{d}^*$
- Continue if warranted $k = k + 1; if \;\; k \le k_{\max} \rightarrow 2.$

Complete analysis job

Mathematical optimization process

Moving asymptotes

$$L_i^{k<3} = d_i^k - (d_{\max} - d_{\min})/2 \quad U_i^{k<3} = d_i^k + (d_{\max} - d_{\min})/2$$

$$L_i^{k\geq 3} = d_i^k - c_k (d_i^{k-1} - L_i^{k-1}) \qquad U_i^{k\geq 3} = d_i^k + c_k (U_i^{k-1} - d_i^{k-1})$$

Convexity factors (simplified)

$$p_i^k = (U_i^k - d_i^k)^2 \frac{\partial f}{\partial d_i^k}; if \quad \frac{\partial f}{\partial d_i^k} \geq 0;$$

$$q_i^k = (d_i^k - L_i^k)^2 \frac{\partial f}{\partial d_i^k}; if \quad \frac{\partial f}{\partial d_i^k} \leq 0;$$

Mathematical problem

$$\hat{f}(\bar{d}^k) = f(\bar{d}^k) + \sum_{i=1}^{N} \left(\frac{p_i^k}{U_i^k - d_i^0} + \frac{q_i^k}{d_i^0 - L_i^k}\right)$$

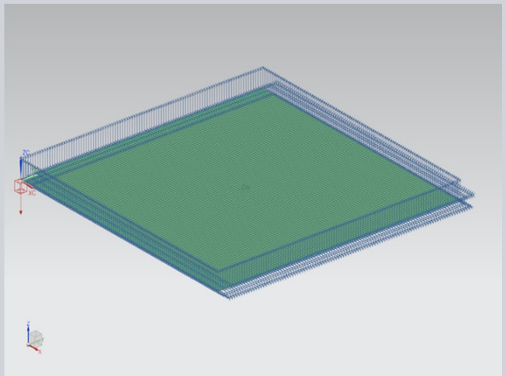
$$\hat{g}_j(\bar{d}^k) = g_j(\bar{d}^k) + \sum_{i=1}^{N} \left(\frac{r_i^k}{U_i^k - d_i^0} + \frac{s_i^k}{d_i^0 - L_i^k}\right)$$

Find and return optimum $\hat{f}(\bar{d}^k) = \min; \hat{g}_j \leq g_j^{\max} \rightarrow \bar{d}^{k+1}, \bar{d}^*$

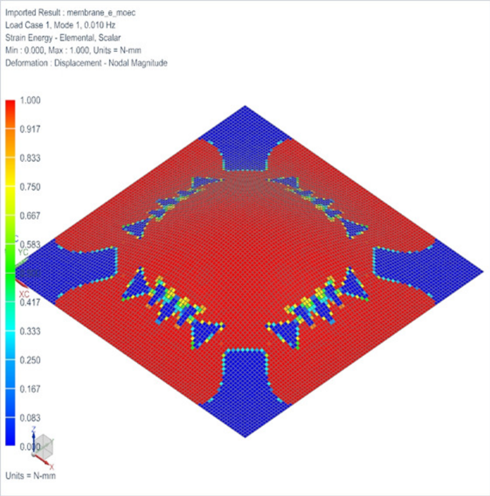
The effect of solution and constraint type

Model statistics

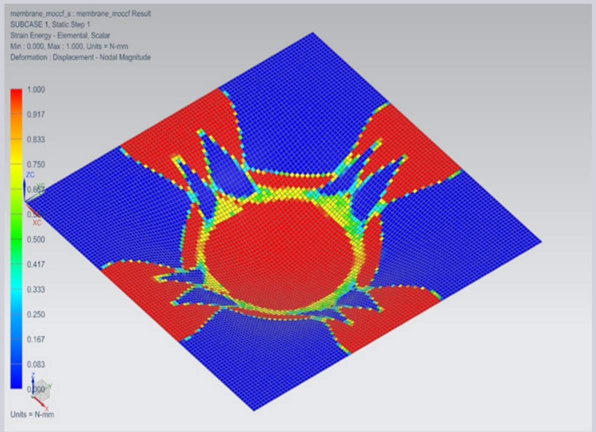
- 10,000 elements (cquad4)
- 10,000 design variables



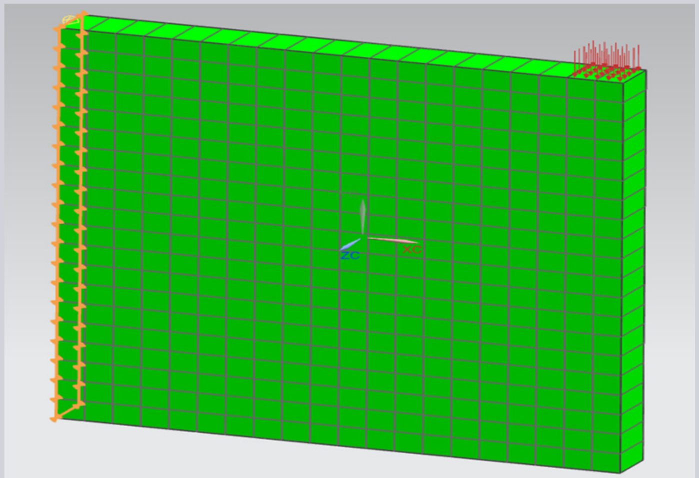
Objective : minimize mass
Constraint : first eigenvalue

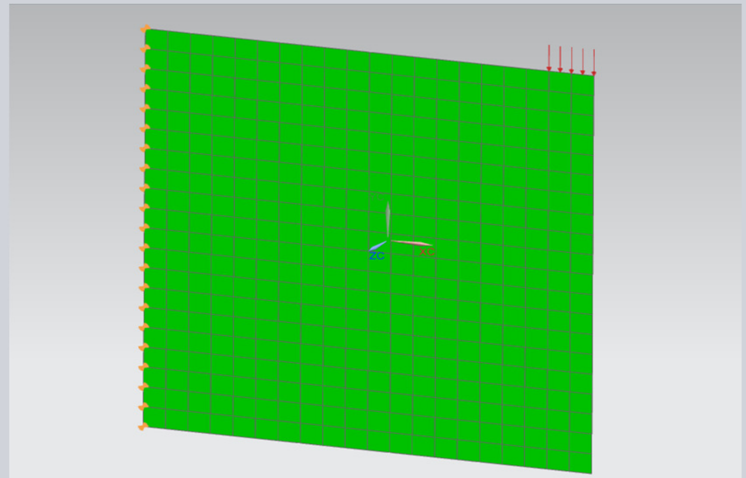


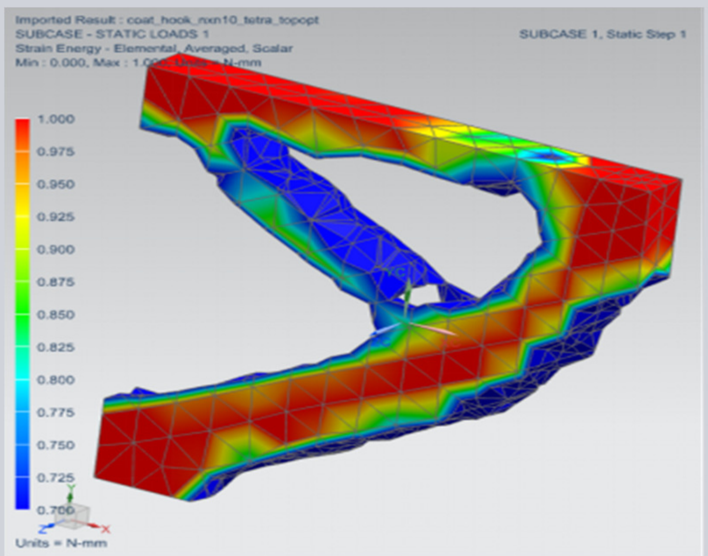
Objective : minimize mass
Constraint : compliance

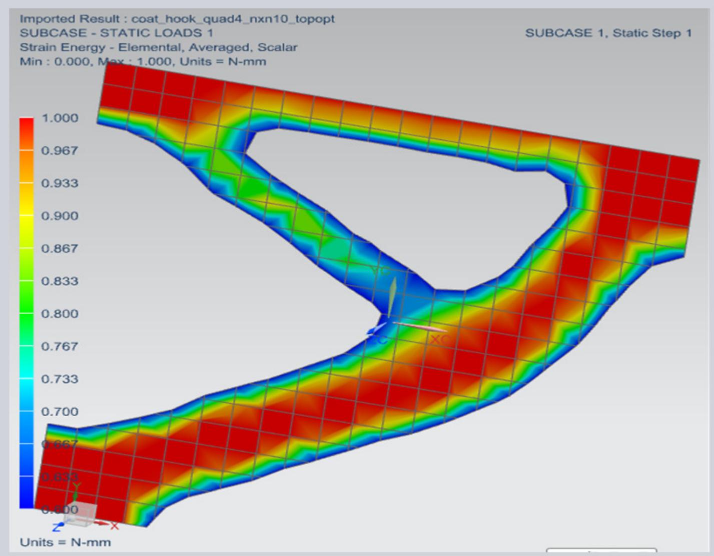


The influence of modeling

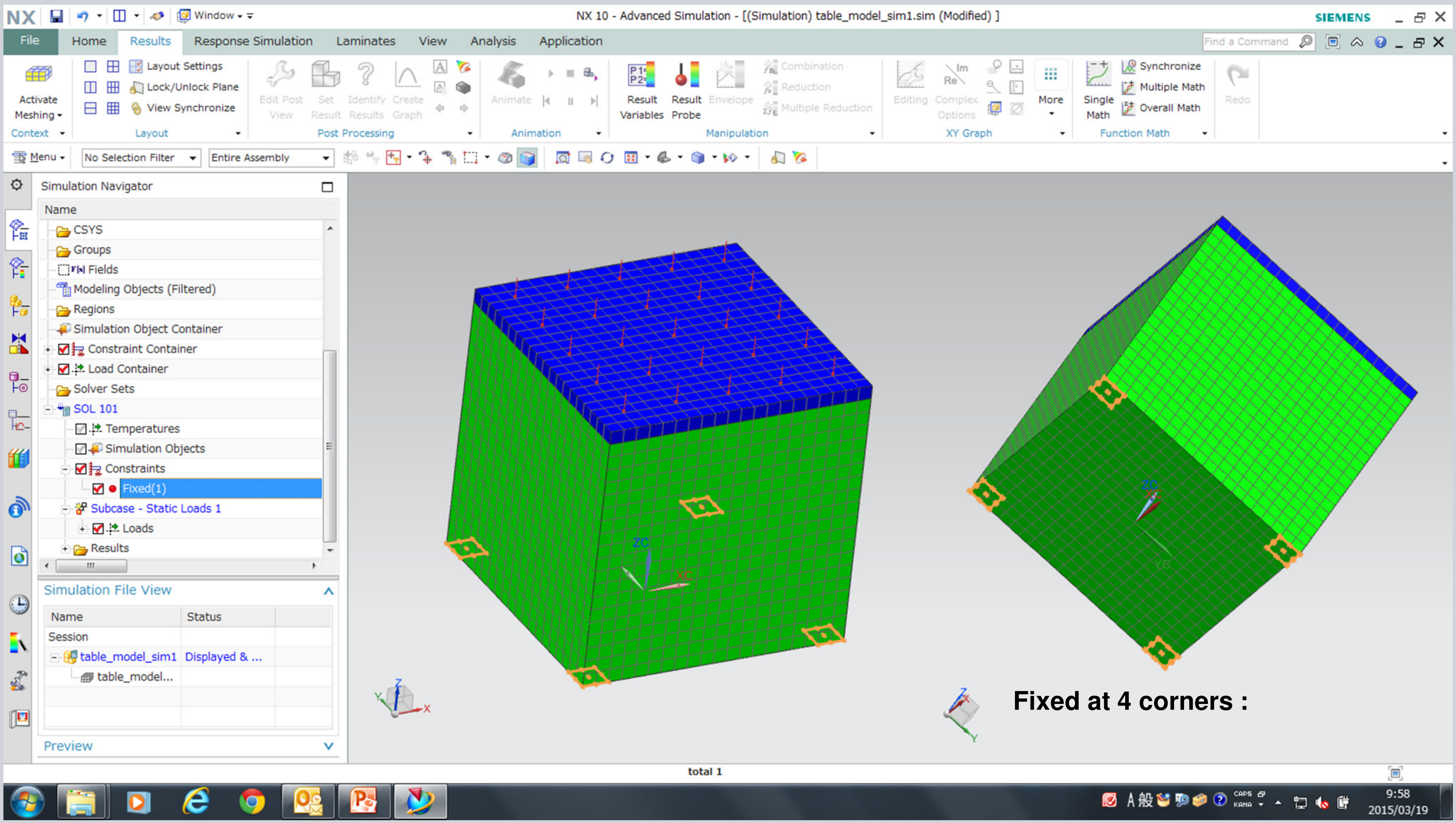




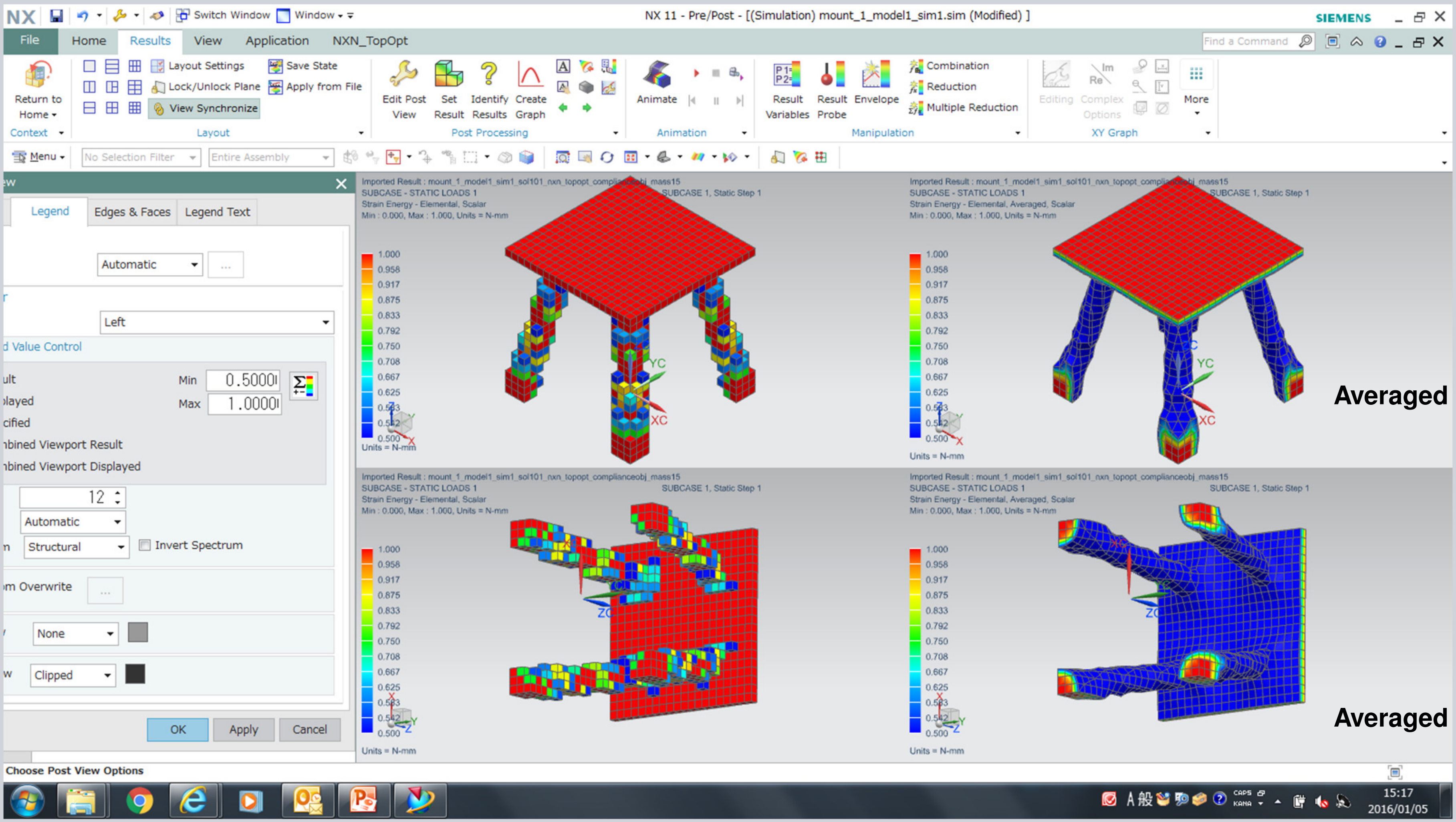




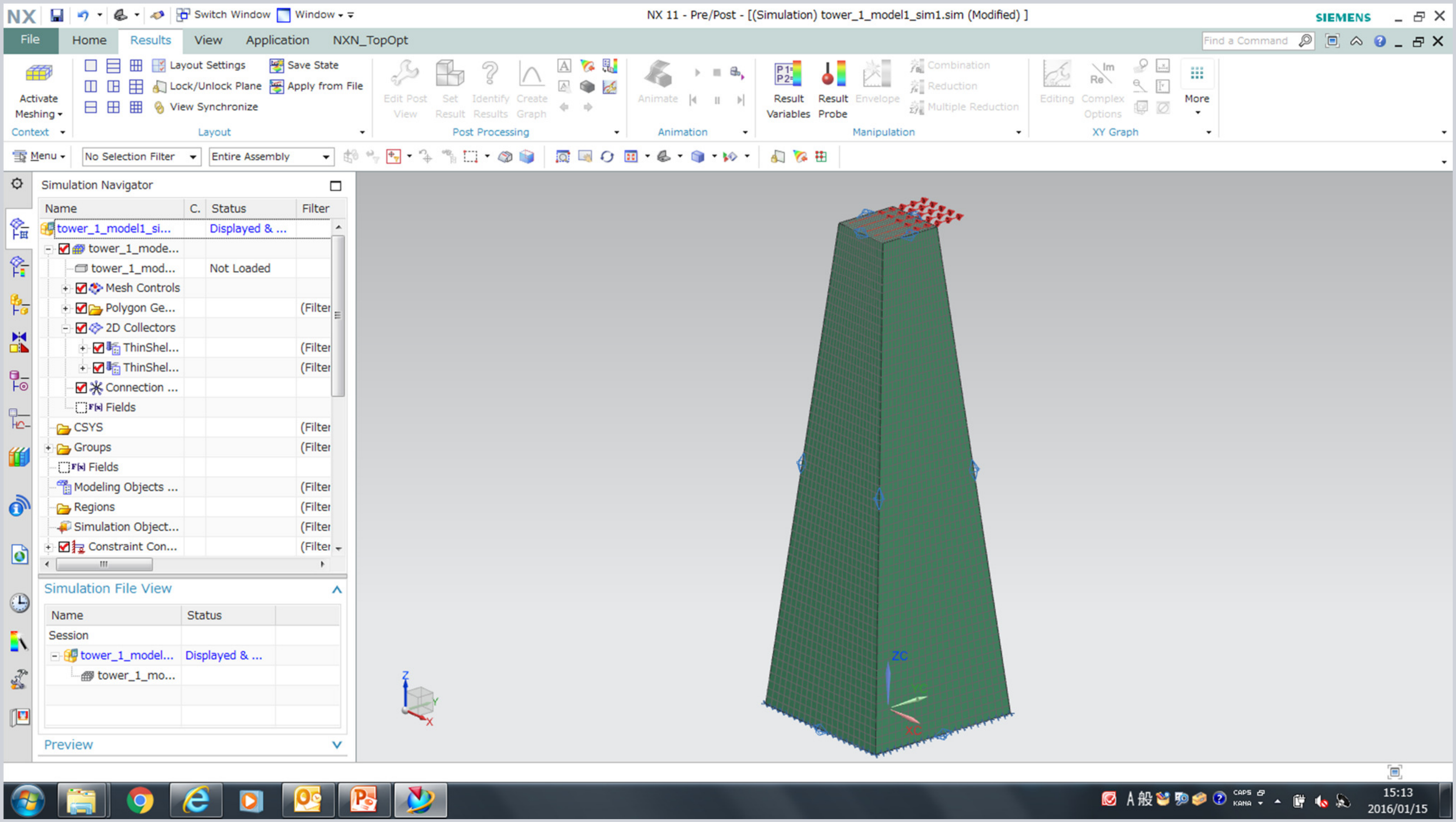
Design a stool



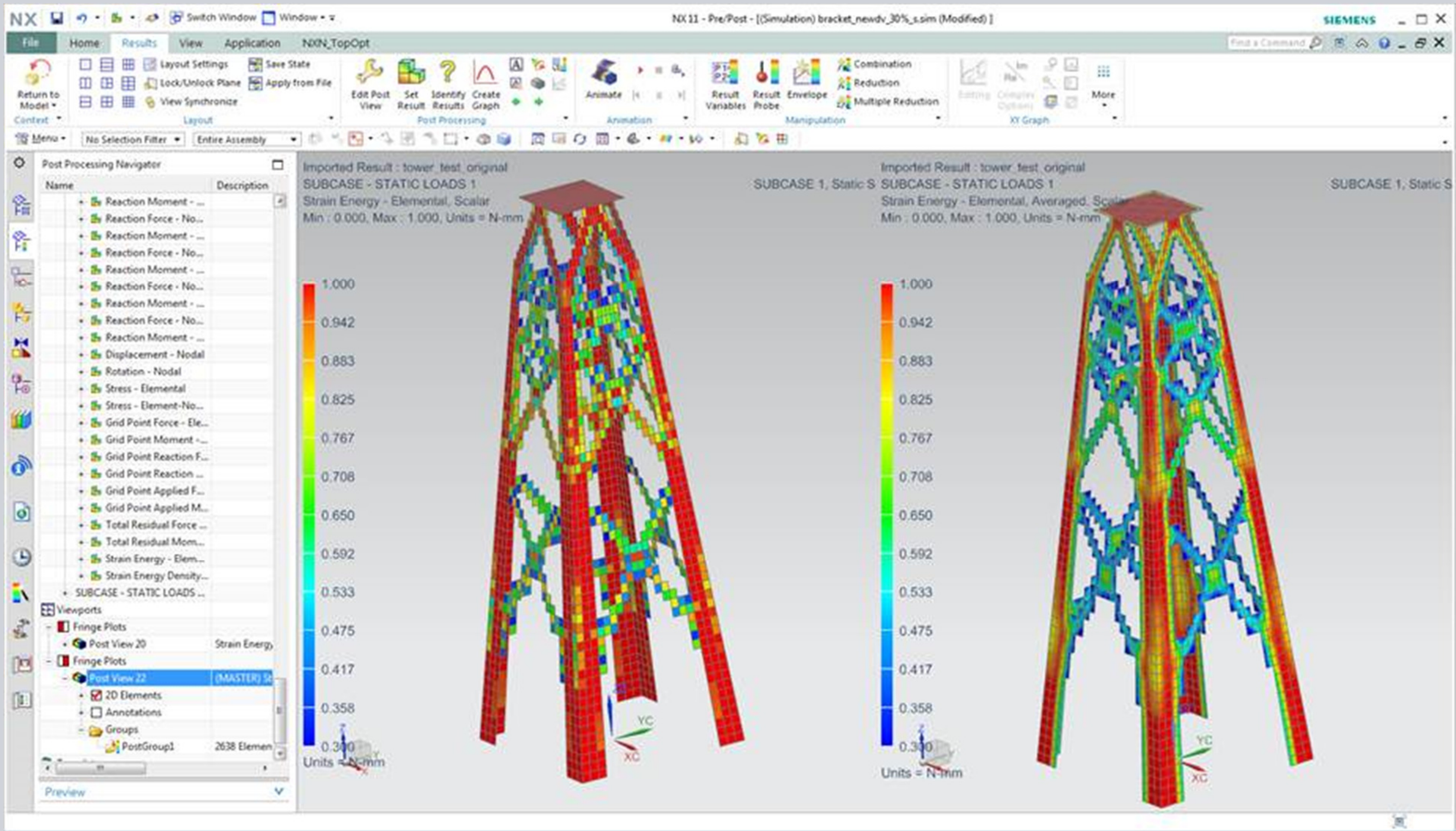
Minimize mass, compliance constraint



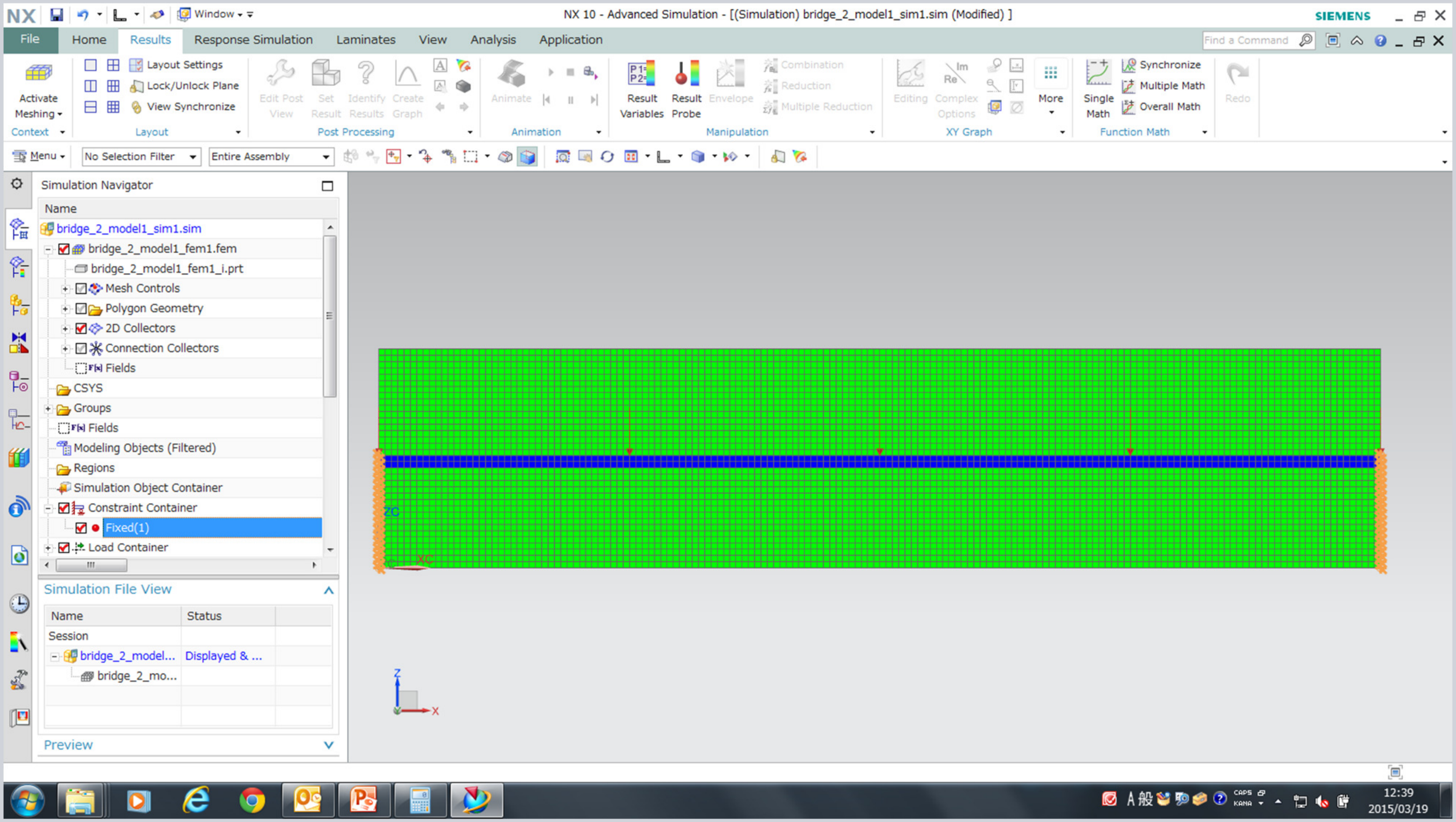
Design a drilling platform



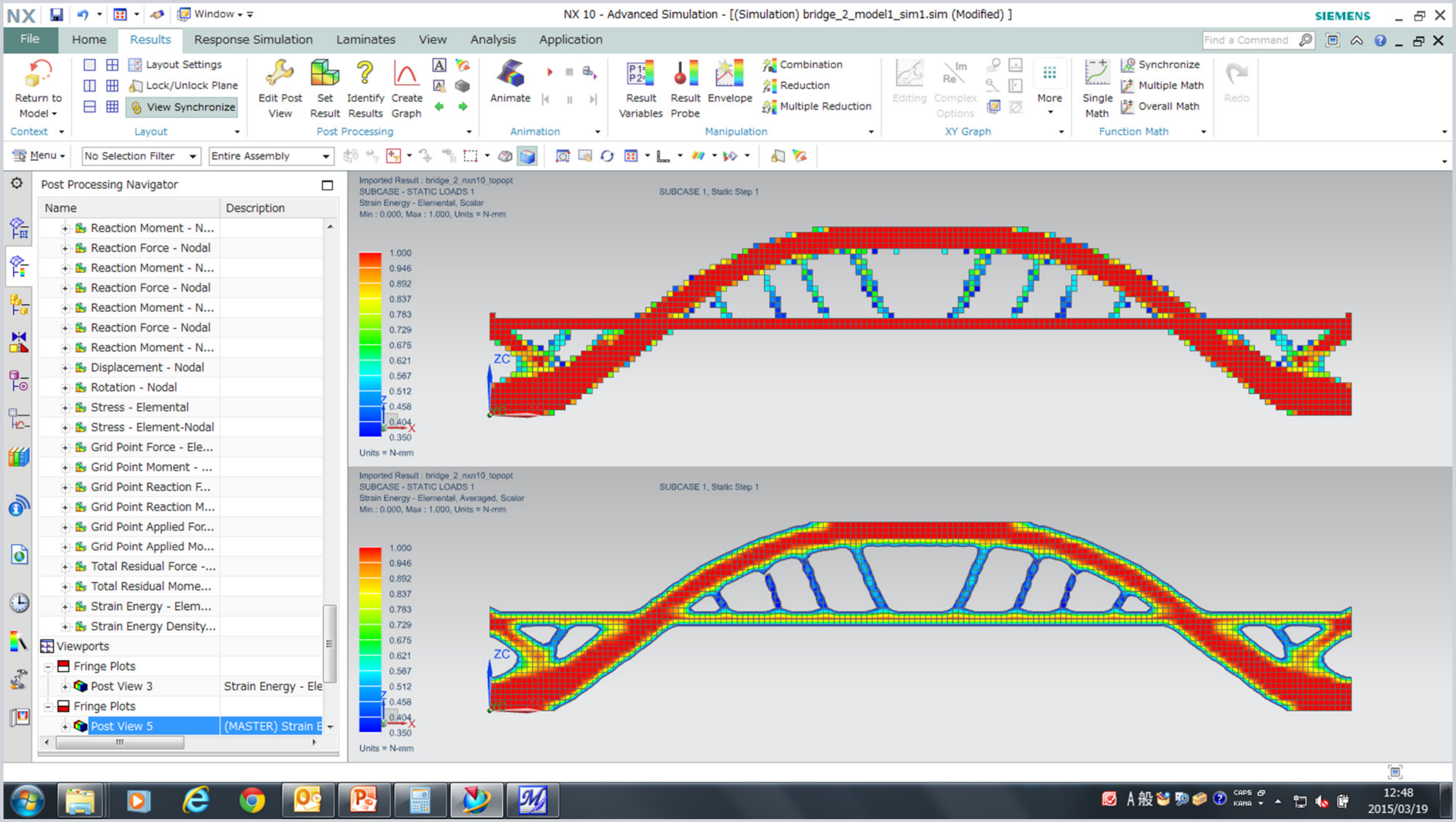
Minimize mass, complaince constraint



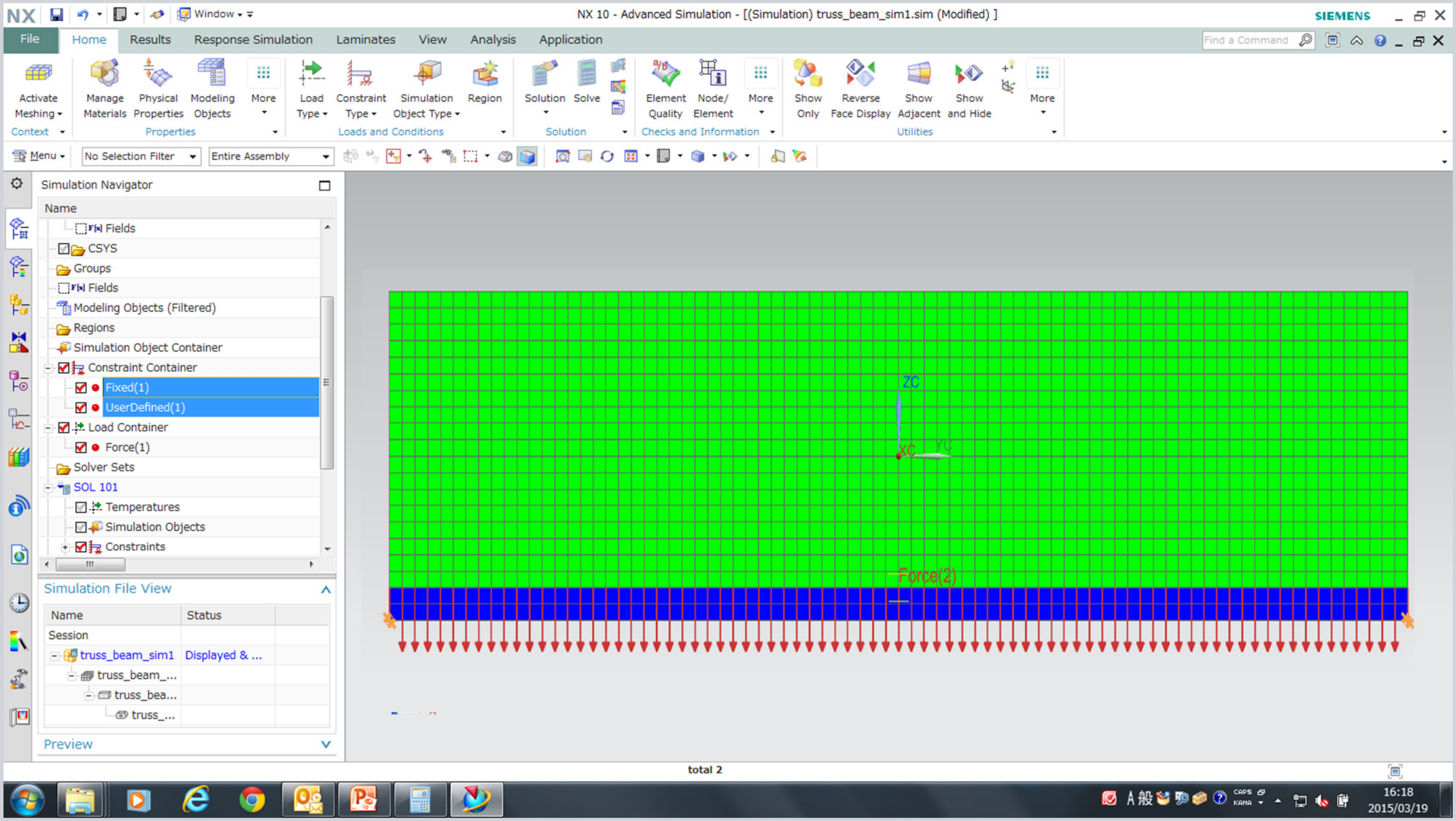
Mid-loaded bridge model design space



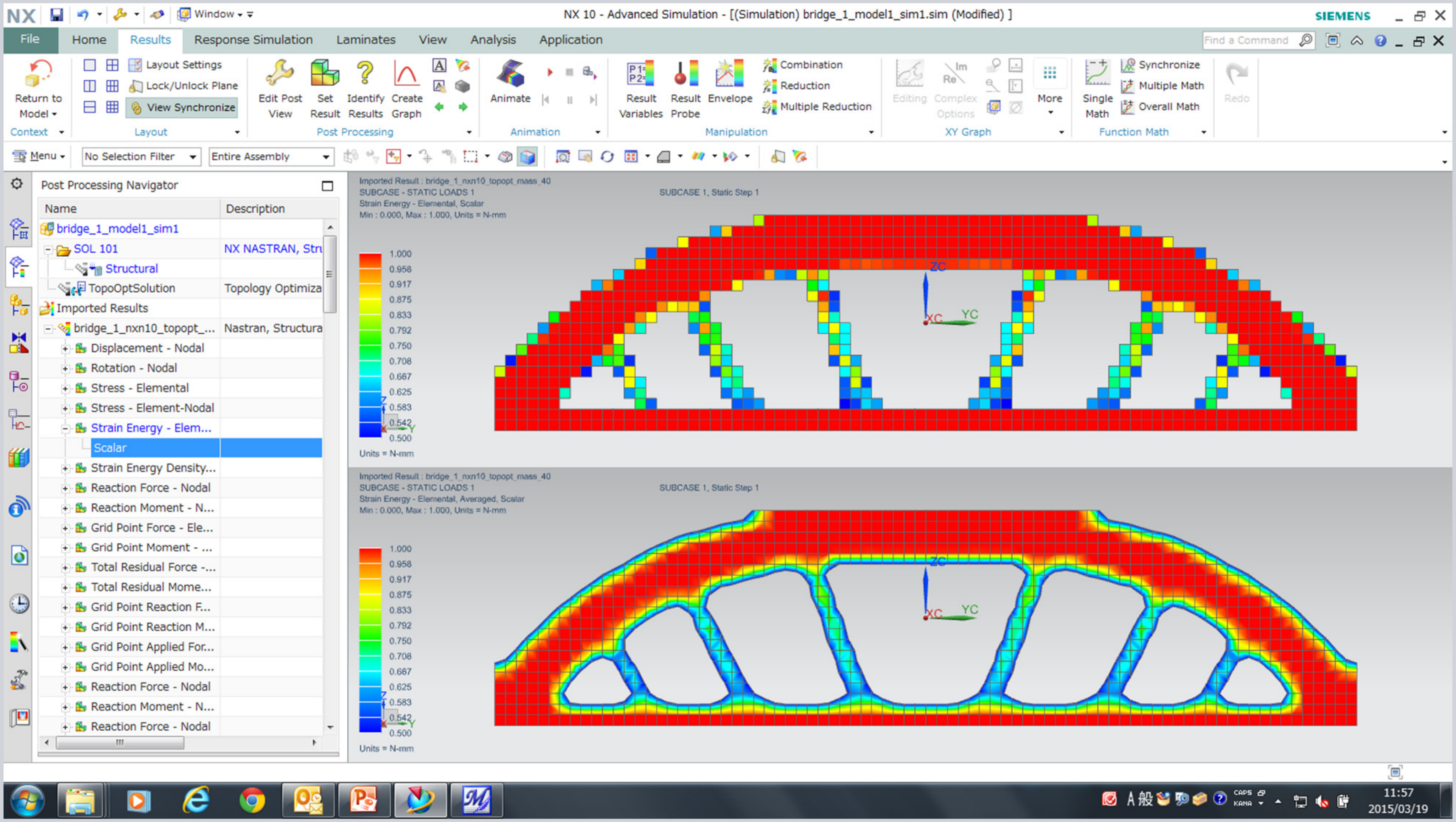
Minimize mass, displacement constraint



Bottom loaded bridge model design space

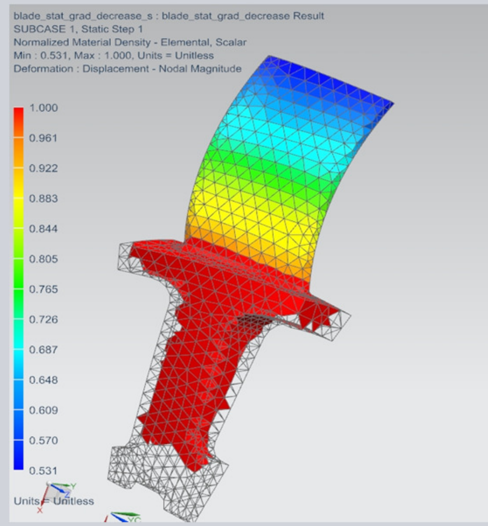


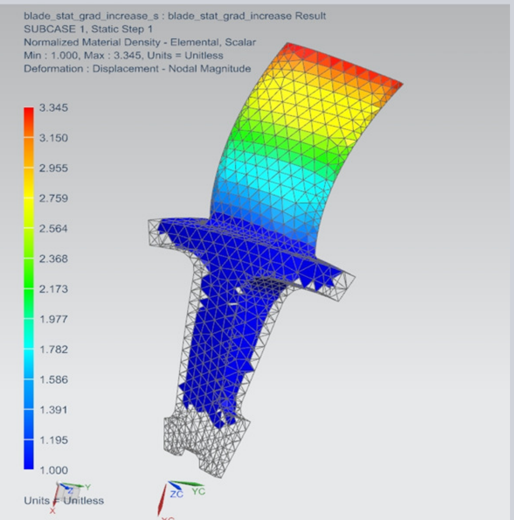
Minimize mass, displacement constraint



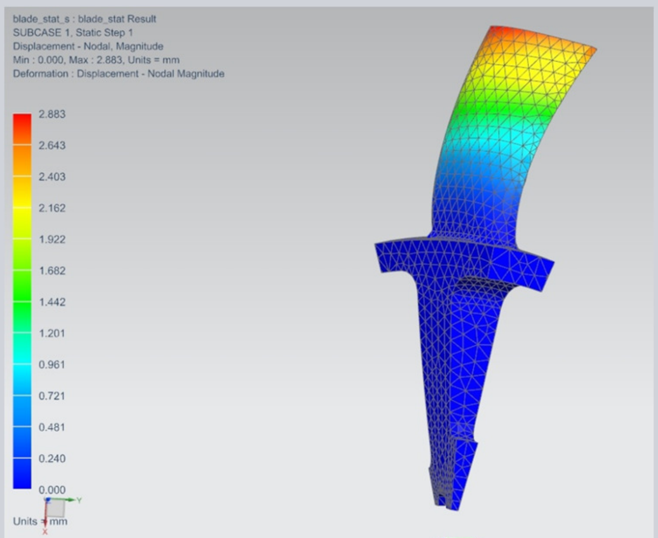
Effect of gradated density to static deformation

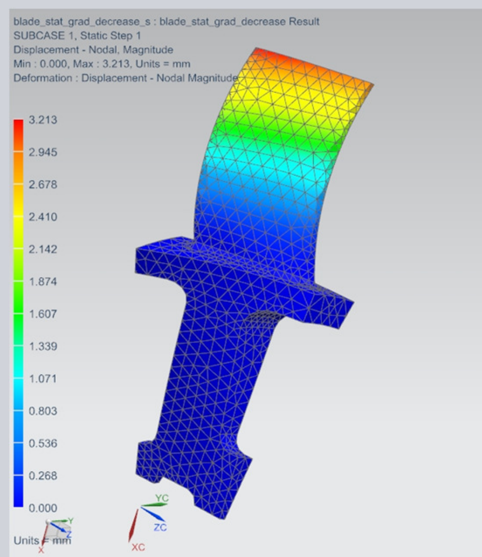
Material gradation

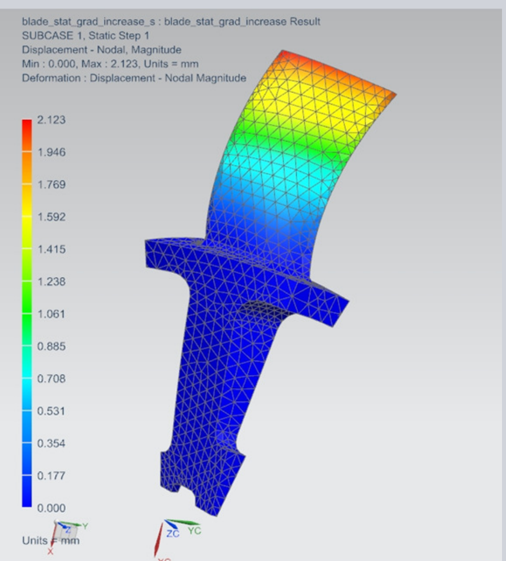




Nominal density deformation

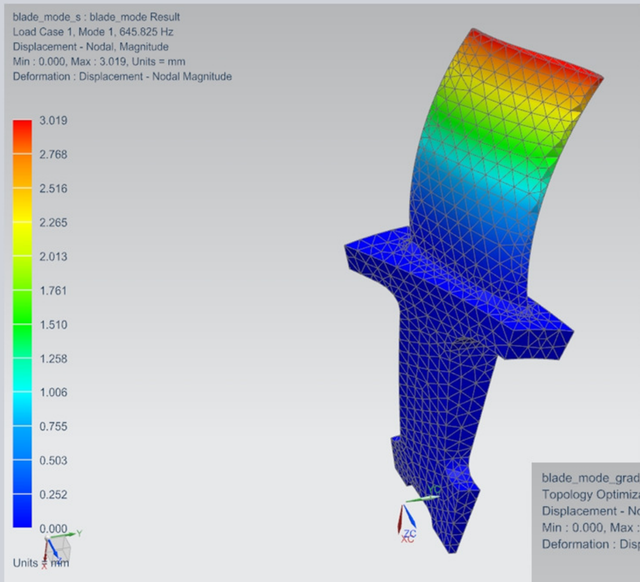




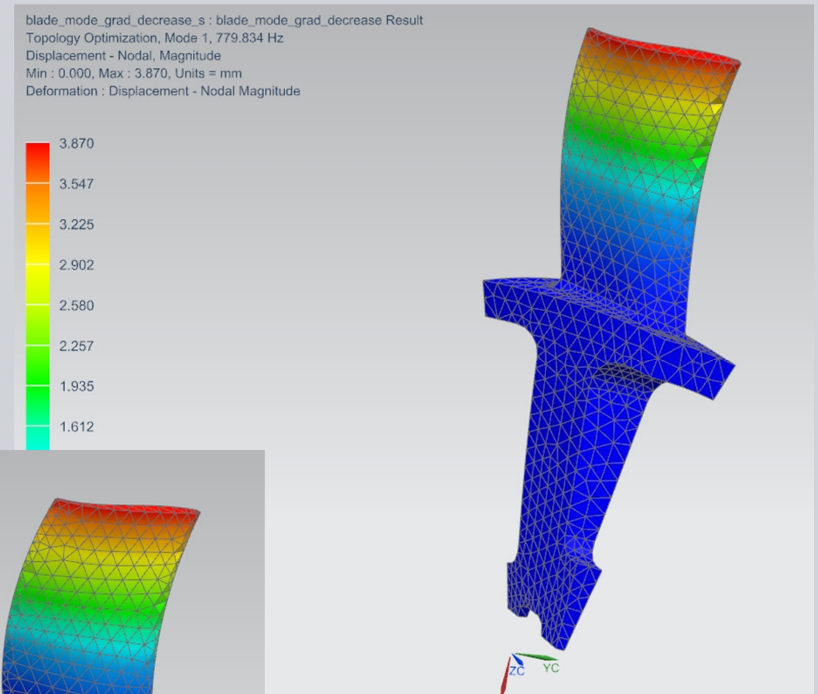


Modified deformations

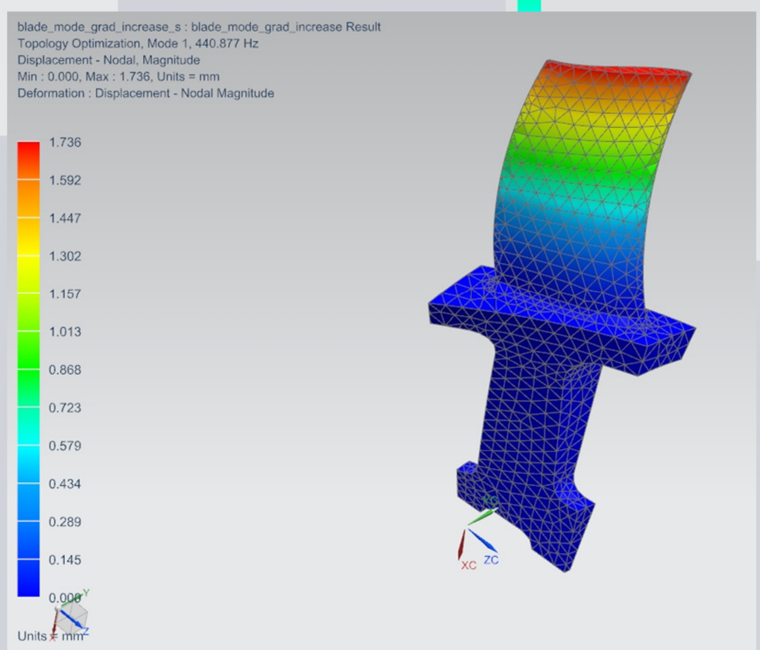
Effect of gradated density to natural frequency



Nominal mode shape

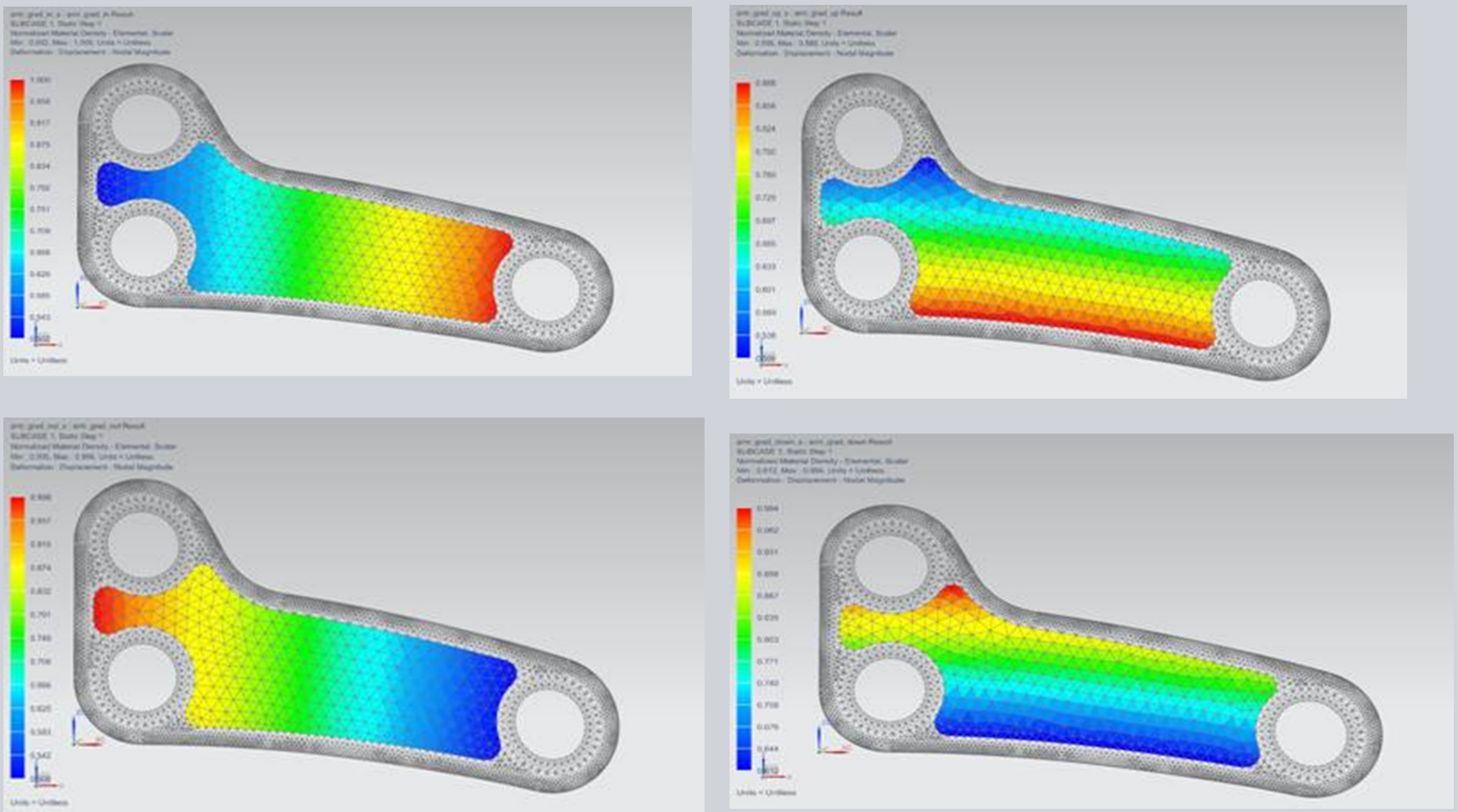


Increasing density inward



Increasing density outward

Density distribution as design tool

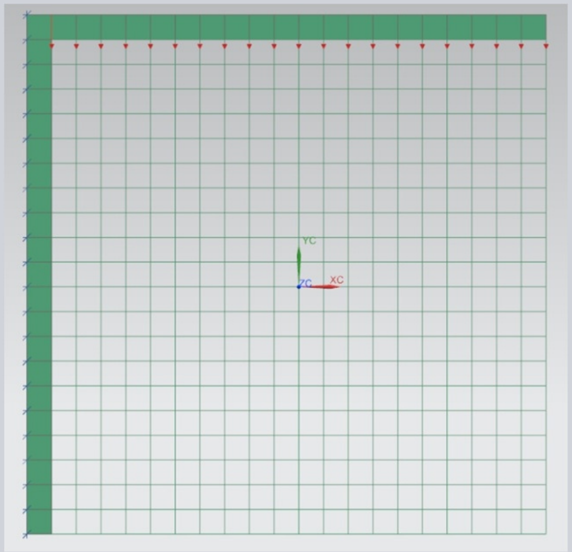


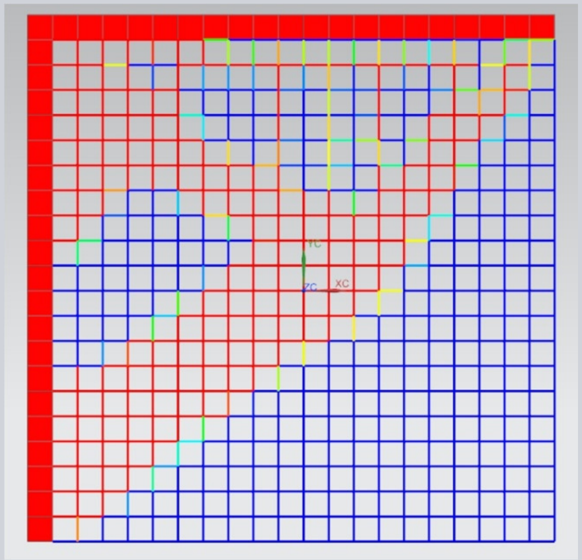
Topology optimization of lattice supported structures

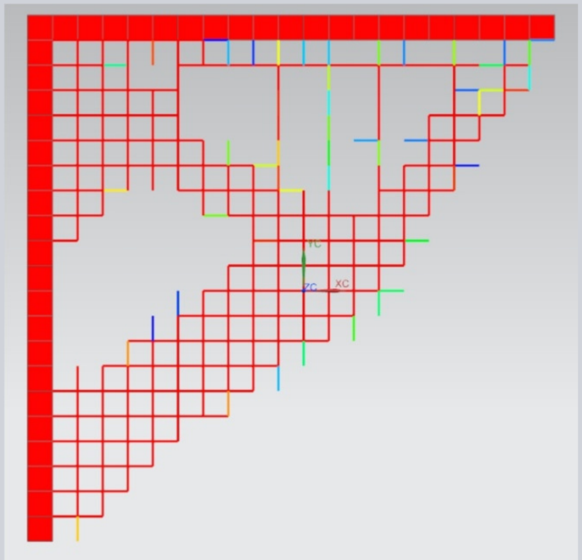
Structure with lattice

Optimization result

Necessary lattice



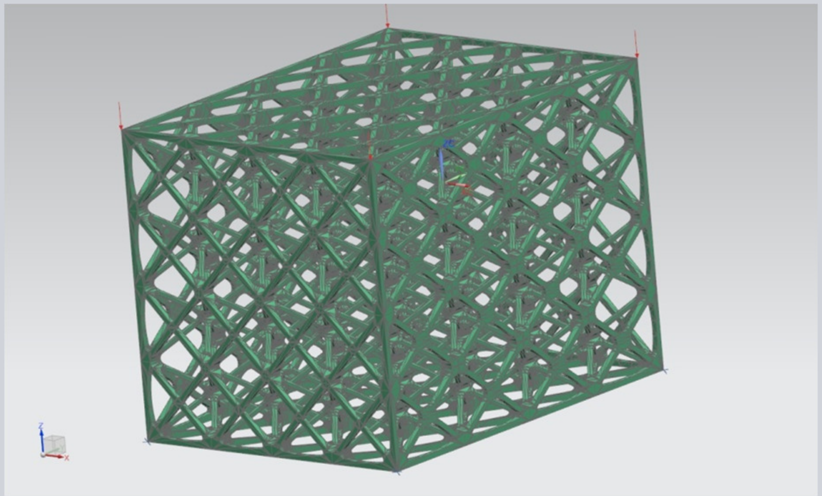


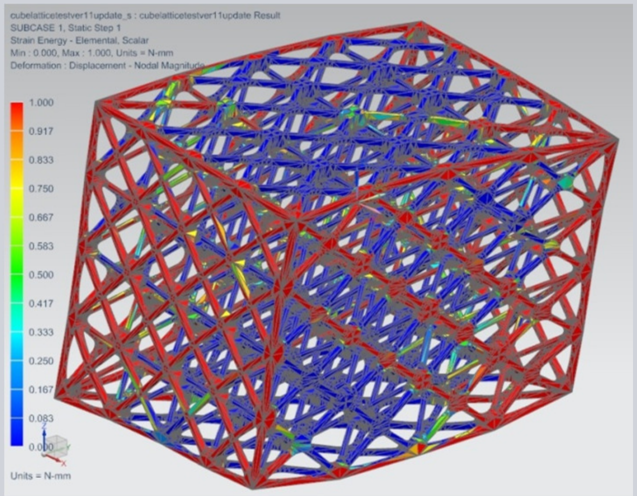


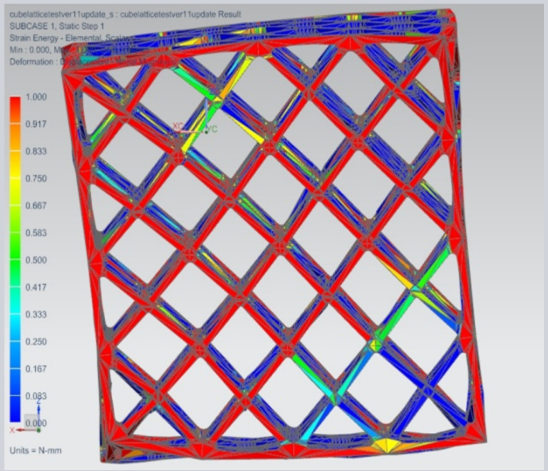
Restricted design space

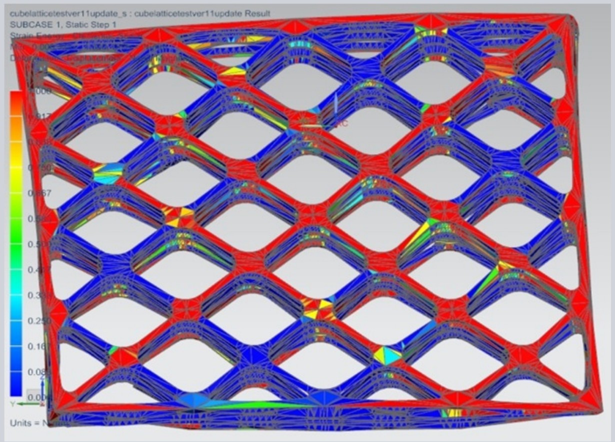
Minimal lattice support

Optimization of additive manufacturing lattice cells









Conclusions

Summary
Topology optimization became an everyday tool in engineering by lowering the need of engineering intuition and finite element method enables its implementation

Topology optimization produces empty spaces inside or a graded density distribution but additive manufacturing is able to build cavities directionally vary the density

References

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Svanberg, K.: A class of globally convergent optimization methods based on conservative covex separable approximations, SIAM J. of Opt., 2002, 12, 555-573